

Bispectrum for non-Gaussian homogeneous and isotropic field on the plane

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Abstract

The object of this paper is to characterize the third order moments (cumulants) and bispectra of a homogeneous isotropic field defined on a plane. We establish a one to one correspondence between the third order cumulants and the bispectra of such a process in terms of Bessel functions.

Dedicated to Lajos Tamásy on the occasion of his 90th birthday.

1 Introduction

In many real applications associated with random fields, the assumption of Gaussianity is unrealistic. For example, consider the data of cosmic microwave background (CMB) anisotropies provided by NASA which is a classic case of non-Gaussianity. [Mar04]. CMB data are given on a surface of a sphere. The assumption of non-Gaussianity is needed in the investigation of the primordial bispectrum given on the whole space, see [VWHK00], [YKW07], [AC12]. In time series analysis the non-Gaussianity has been well studied [SRG80], [SRG84], [Hin82], [TM98], [Ter99]. It is known that for a Gaussian time series the bispectrum and all higher order spectra are zero, and equally well known is the fact that for a non Gaussian process defined on a real line the higher order cumulant spectra and higher order cumulants uniquely determine each other [Bri01], similar questions has been considered recently for fields on spheres [MP10], [Ter13]. No such results are available and well known for a homogeneous isotropic field defined on a plane. The Gaussian homogeneous and isotropic fields are well studied see [Yag87] and references therein.

In this paper we first consider a homogeneous field and define the homogeneous and isotropic fields for general not necessarily Gaussian fields. The second order properties are given which serves as an example of the method we use for obtaining an expression for the third order spectra. The main object of this paper is to describe the unique relation between the third order covariances (bicovariances) and the bispectrum of a homogeneous and isotropic field on the plane.

1.1 Non-Gaussian homogeneous and isotropic field on the plane

We consider a homogeneous real valued stochastic field $X(\underline{x})$ on \mathbb{R}^2 with $X(\underline{x}) = 0$. Let us suppose that $X(\underline{x})$ is continuous (in mean square sense), its spectral representation is

$$X(\underline{x}) = \int_{\mathbb{R}^2} e^{i\underline{x} \cdot \underline{\omega}} Z(d\underline{\omega}), \quad \underline{\omega}, \underline{x} \in \mathbb{R}^2,$$

with a finite spectral measure

$$E |Z(d\omega)|^2 = F_0(d\omega).$$

By homogeneity we mean (in strict sense) the distribution of $X(\underline{x})$ is translation invariant and all necessary moments exist, see [Yag87] for details. Rewrite $X(\underline{x})$ in term of polar coordinates

$$\begin{aligned} X(r, \varphi) &= \int_0^\infty \int_0^{2\pi} e^{i\rho r \cos(\varphi-\eta)} Z(\rho d\rho d\eta) \\ &= \sum_{\ell=-\infty}^\infty e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho) \end{aligned} \quad (1)$$

where J_ℓ denotes the Bessel function of the first kind, $\underline{x} = (r, \varphi)$, $\underline{\omega} = (\rho, \eta)$ are polar coordinates, $r = |\underline{x}| = \sqrt{x_1^2 + x_2^2}$, $\rho = |\underline{\omega}|$, $\underline{x} \cdot \underline{\omega} = r\rho \cos(\varphi - \eta)$ and in the above we used the following notation

$$Z_\ell(\rho d\rho) = \int_0^{2\pi} i^\ell e^{-i\ell\eta} Z(\rho d\rho d\eta).$$

The representation (1) will be an orthogonal representation if we assume that $F_0(d\omega)$ is isotropic (invariant under rotations) $F_0(d\omega) = E |Z(d\omega)|^2 = E |Z(\rho d\rho d\eta)|^2 = F(\rho d\rho) d\eta$. Under this assumption we consider the Jacobi-Anger expansion

$$e^{i\rho r \cos(\varphi-\eta)} = \sum_{\ell=-\infty}^\infty i^\ell J_\ell(\rho r) e^{i\ell(\varphi-\eta)}, \quad (2)$$

and substitute this into the spectral representation of $X(r, \varphi)$. The stochastic spectral measures $Z_\ell(\rho d\rho)$ defined above has the following property

$$\begin{aligned} Z_{-\ell}(\rho d\rho) &= \int_0^{2\pi} i^{-\ell} e^{-i\ell\eta} Z(\rho d\rho d\eta) \\ &= \int_0^{2\pi} i^\ell e^{i\ell\eta} \overline{Z(\rho d\rho d\eta)} \\ &= (-1)^\ell \overline{Z_\ell(\rho d\rho)}, \end{aligned}$$

since $e^{i\underline{x} \cdot \underline{\omega}} Z(d\omega) = e^{-i\underline{x} \cdot \underline{\omega}} \overline{Z(d\omega)}$, and this follows from the fact that the field $X(\underline{x})$ is real valued. Moreover it is orthogonal

$$\begin{aligned} \text{Cov}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)) &= \int_0^{2\pi} i^{(\ell_1-\ell_2)} e^{-i(\ell_2-\ell_1)\eta} F_0(d\eta \rho d\rho) \\ &= \delta_{\ell_1-\ell_2} 2\pi F(\rho d\rho), \end{aligned}$$

where $\delta_{\ell_1-\ell_2}$ is the Kronecker delta. The well known property of Bessel functions $J_\ell = (-1)^\ell J_{-\ell}$ implies that in the representation (1) the ℓ^{th} term and the $-\ell^{\text{th}}$ term are conjugate to each other. Note that the spectral measure of the stochastic measure $Z_\ell(\rho d\rho)$ does not depend on ℓ . We shall assume in particular cases that $F(\rho d\rho)$ is absolutely continuous, i.e. $F(\rho d\rho) = \sigma^2 |A(\rho)|^2 \rho d\rho$, here $\sigma^2 |A(\rho)|^2$ is usually known as the second order spectrum. In other words the representation (1) has the particular form

$$X(r, \varphi) = \sum_{\ell=-\infty}^\infty e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) A(\rho) W_\ell(\rho d\rho),$$

such that $W_\ell(\rho d\rho)$ is a series of white noise measures with constant spectrum, hence

$$\text{Cov}(W_{\ell_1}(\rho_1 d\rho_1), W_{\ell_2}(\rho_2 d\rho_2)) = \delta_{\ell_1-\ell_2} \sigma^2 \rho d\rho.$$

The general theorem of Yadrenko ([Yad83] Theorem 1. pp.5) on the spectral representation of a homogeneous and isotropic field $X(r, \varphi)$ gives the representation (1) with real valued stochastic spectral measures $Z_\ell(\rho d\rho)$ constructed directly from the field $X(r, \varphi)$ itself.

2 Isotropy on the plane

We consider rotations about the origin of the coordinate system. Under a rotation (passive) $g \in SO(2)$, by which we mean a rotation when vectors remain fixed, but the point it defines is given by a new set of coordinates. A rotation g is characterized by an angle γ and by the rotation matrix

$$g = \begin{bmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{bmatrix}.$$

If $\underline{x} \in \mathbb{R}^2$ is given in polar coordinates $\underline{x} = (r, \varphi)$, then $g\underline{x} = (r, \varphi - \gamma)$, and as usual the operator $\Lambda(g)$ acts on functions $f(r, \varphi)$, such that $\Lambda(g)f(r, \varphi) = f(g^{-1}(r, \varphi)) = f(r, \varphi + \gamma)$.

The isotropy usually is defined through the invariance of the covariance structure. This is satisfactory for Gaussian cases but for non-Gaussian fields we need invariance of higher order cumulants as well. We need a stronger definition to achieve a similar invariance to be able to define third order spectra, which we will propose below.

Definition 1 *A homogeneous stochastic field $X(\underline{x})$ is strictly isotropic if all finite dimensional distributions of $X(\underline{x})$ are invariant under rotations.*

If the homogeneous field $X(\underline{x})$ is Gaussian, then the isotropy of the spectral measure $F_0(d\underline{\omega})$, i.e. in polar coordinates $F_0(d\underline{\omega}) = F(\rho d\rho) d\eta / (2\pi)$, implies

$$\text{Cov}(\Lambda(g)X(\underline{x}_1), \Lambda(g)X(\underline{x}_2)) = \text{Cov}(X(\underline{x}_1), X(\underline{x}_2)),$$

for each $\underline{x}_1, \underline{x}_2$ and for every $g \in SO(2)$. That is the distribution of a Gaussian isotropic field is invariant under rotations. The definition given above is a generalization of this property which will be used later. If the field $X(\underline{x})$ is non-Gaussian and if we assume the existence of all moments then isotropy follows and followed by that all higher order moments are also invariant under rotation. Let us consider a homogeneous and isotropic stochastic field $X(\underline{x}) = X(r, \varphi)$, ($r > 0, \varphi \in [0, 2\pi)$) on the plane

$$X(r, \varphi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^{\infty} J_{\ell}(\rho r) Z_{\ell}(\rho d\rho),$$

where Z_{ℓ} is an array of measures, orthogonal to each other

$$\text{Cov}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)) = \delta_{\ell_1 - \ell_2} 2\pi F(\rho d\rho).$$

In this representation $e^{i\ell\varphi}$ plays a role of spherical harmonics of degree ℓ with complex values on the plane. In this way an isotropic random field $X(\underline{x})$ can be decomposed into a countable number of mutually uncorrelated spectral measures with a one dimensional parameter.

The rotation g takes effect on the 'spherical harmonics' $e^{i\ell m\varphi}$ ($m = \pm 1$), as $\Lambda(g)e^{i\ell m\varphi} = e^{i\ell m(\varphi + \gamma)} = e^{i\ell m\gamma} e^{i\ell m\varphi}$, since the 'spherical harmonics' $e^{i\ell m\varphi}$ can be considered as a function of φ . The isotropy of

$$X(r, \varphi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^{\infty} J_{\ell}(\rho r) Z_{\ell}(\rho d\rho),$$

implies that the distribution of $X(r, \varphi)$ does not change under rotations $g \in SO(2)$. Consider

$$\begin{aligned} \Lambda(g)X(r, \varphi) &= \sum_{\ell=-\infty}^{\infty} e^{i\ell(\varphi + \gamma)} \int_0^{\infty} J_{\ell}(\rho r) Z_{\ell}(\rho d\rho) \\ &= \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^{\infty} J_{\ell}(\rho r) e^{i\ell\gamma} Z_{\ell}(\rho d\rho) \\ &= \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^{\infty} J_{\ell}(\rho r) Z_{\ell}(\rho d\rho), \end{aligned}$$

hence the distribution of $Z_\ell(\rho d\rho)$ and $e^{i\ell\gamma}Z_\ell(\rho d\rho)$ should be the same. Now it is evident that for a Gaussian random field $X(r, \varphi)$, the necessary and sufficient condition of isotropy is the $Z_\ell(\rho d\rho)$ be independent measures. Indeed under isotropy assumption we have

$$\text{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)) = e^{i(\ell_1 + \ell_2)\gamma} \text{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)),$$

for each γ , hence either $\ell_1 + \ell_2 = 0$, or $\text{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)) = 0$, therefore

$$\text{Cov}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2)) = 0,$$

unless $\ell_1 = \ell_2$.

In general, we have under isotropy assumption

$$\text{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), \dots, Z_{\ell_p}(\rho_p d\rho_p)) = e^{i(\ell_1 + \ell_2 + \dots + \ell_p)\gamma} \text{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), \dots, Z_{\ell_p}(\rho_p d\rho_p)),$$

that is if $\ell_1 + \ell_2 + \dots + \ell_p = 0$, or $\text{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2), \dots, Z_{\ell_p}(\rho_p d\rho_p)) = 0$ should be fulfilled. In turn, if this assumption is satisfied then the cumulants $\text{Cum}(Z_{\ell_1}(\rho_1 d\rho_1), \dots, Z_{\ell_p}(\rho_p d\rho_p))$ are invariant under rotations and if in addition the distributions of $X(r, \varphi)$ are determined by the moments then the field is isotropic.

2.1 Spectrum

Briefly, we consider the second order structure of a homogeneous and isotropic stochastic field which is well studied for Gaussian cases. For notational convenience let us denote the stochastic process

$$z_\ell(r) = \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho),$$

and consider the covariance

$$\begin{aligned} \text{Cov}(X(\underline{x}), X(\underline{y})) &= \text{Cov}\left(\sum_{\ell_1=-\infty}^{\ell_1=\infty} e^{i\ell_1\varphi_1} z_{\ell_1}(r_1), \sum_{\ell_2=-\infty}^{\ell_2=\infty} e^{i\ell_2\varphi_2} z_{\ell_2}(r_2)\right) \\ &= 2\pi \int_0^\infty \sum_{\ell=-\infty}^{\infty} e^{i\ell(\varphi_1 - \varphi_2)} J_\ell(\rho r_1) J_\ell(\rho r_2) F(\rho d\rho) \\ &= 2\pi \int_0^\infty J_0(\rho r) F(\rho d\rho), \end{aligned}$$

where $r = |\underline{x} - \underline{y}|$. In arriving at the above formula we used the addition formula of Bessel functions, see [EMOT54] Tom2, Ch7, 7.6.2.(6), [Yad83]. Now one may derive the same result using the properties of homogeneity and isotropy. We have in general

$$X(r, \varphi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho),$$

and in a special case

$$X(rN) = \sum_{\ell=-\infty}^{\infty} i^\ell \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho), \quad (3)$$

$$\begin{aligned} X(\underline{Q}) &= \int_{\mathbb{R}^2} Z(d\underline{\omega}) \\ &= \int_0^\infty Z_0(\rho d\rho), \end{aligned} \quad (4)$$

where the 'North pole' $N = (0, 1)$. Let $r = |\underline{x} - \underline{y}|$, $\mathcal{C}_2(r) = \text{Cov}(X(\underline{x}), X(\underline{y}))$, and use the invariance under translation and rotation to obtain

$$\begin{aligned}\text{Cov}(X(\underline{x}), X(\underline{y})) &= \text{Cov}(X(\underline{x} - \underline{y}), X(0)) \\ &= \text{Cov}(X(rN), X(0)) \\ &= 2\pi \int_0^\infty J_0(\rho r) F(\rho d\rho).\end{aligned}$$

The above shows one to one correspondence between the second order covariance and its spectral density function, see [Yad83]. In particular for absolutely continuous spectral measure $F(\rho d\rho) = \sigma^2 |A(\rho)|^2 \rho d\rho$ we have

$$\mathcal{C}_2(r) = 2\pi \int_0^\infty J_0(\rho r) \sigma^2 |A(\rho)|^2 \rho d\rho,$$

in turn

$$\sigma^2 |A(\rho)|^2 = \frac{1}{2\pi} \int_0^\infty J_0(\rho r) \mathcal{C}_2(r) r dr,$$

in case that the integral exists, see [Bri74], [Yag87], for instance. The above property of Hankel transform used above is based on the following property of Bessel functions

$$\int_0^\infty J_0(\rho r) J_0(\kappa r) r dr = \frac{\delta(\rho - \kappa)}{\rho},$$

where $\delta(\rho - \kappa)$ denotes the Dirac 'function', more precisely $\delta(\rho - \kappa)$ is a distribution, see [AW01] Sect 11.

3 Bispectrum

The third order structure of a homogeneous and isotropic stochastic field $X(\underline{x})$ is described by either the third order cumulants (bicoariances) in spatial domain or the bispectrum in frequency domain. Using the spectral representations, we obtain the third order cumulants (central moments) and it is given by

$$\text{Cum}(X(\underline{x}_1), X(\underline{x}_2), X(\underline{x}_3)) = \iiint_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} e^{i(\Sigma_1^3 \underline{x}_k \cdot \underline{\omega}_k)} S_3(\underline{\omega}_1, \underline{\omega}_2, \underline{\omega}_3) \delta(\Sigma_1^3 \underline{\omega}_k) \prod_{k=1}^3 d\underline{\omega}_k,$$

under isotropy for each $g \in SO(2)$

$$\begin{aligned}\text{Cum}(X(g\underline{x}_1), X(g\underline{x}_2), X(g\underline{x}_3)) &= \iiint_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} e^{i(\Sigma_1^3 \underline{x}_k \cdot \underline{\omega}_k)} S_3(g\underline{\omega}_1, g\underline{\omega}_2, g\underline{\omega}_3) \delta(\Sigma_1^3 \underline{\omega}_k) \prod_{k=1}^3 d\underline{\omega}_k \\ &= \text{Cum}(X(\underline{x}_1), X(\underline{x}_2), X(\underline{x}_3))\end{aligned}$$

hence $S_3(g\underline{\omega}_1, g\underline{\omega}_2, g\underline{\omega}_3) = S_3(\underline{\omega}_1, \underline{\omega}_2, \underline{\omega}_3) = S_3(\rho_1, \rho_2, \rho_3)$. Now we apply the invariance of the bicoariance

$$\begin{aligned}\text{Cum}(X(\underline{x}_1), X(\underline{x}_2), X(\underline{x}_3)) &= \text{Cum}(X(0), X(\underline{x}_2 - \underline{x}_1), X(\underline{x}_3 - \underline{x}_1)) \\ &= \text{Cum}(X(0), X(|\underline{x}_2 - \underline{x}_1| N), X(g(\underline{x}_3 - \underline{x}_1)))\end{aligned}$$

where g denotes the rotation carrying $\underline{x}_2 - \underline{x}_1$ into the North pole. We use the particular representations (4) and (3) and get

$$\begin{aligned} \text{Cum}(X(0), X(r_2 N), X(\underline{x}_3)) &= \sum_{\ell_2, \ell_3 = -\infty}^{\infty} \int_0^{\infty} \int \int i^{\ell_2} e^{i\ell_3 \varphi_3} J_{\ell_2}(\rho_2 r_2) J_{\ell_3}(\rho_3 r_3) \\ &\quad \times \text{Cum}(Z_0(\rho_1 d\rho_1), Z_{\ell_2}(\rho_2 d\rho_2), Z_{\ell_3}(\rho_3 d\rho_3)) \\ &= \sum_{\ell = -\infty}^{\infty} \int_0^{\infty} \int \int e^{i\ell \varphi} J_{\ell}(\rho_2 r_2) J_{-\ell}(\rho_3 r_3) \\ &\quad \times \text{Cum}(Z_0(\rho_1 d\rho_1), Z_{\ell}(\rho_2 d\rho_2), Z_{-\ell}(\rho_3 d\rho_3)) \end{aligned}$$

where $\varphi = \pi/2 - \varphi_3$, is the angle between N and \underline{x}_3 . The third order cumulant of the stochastic spectral measure $Z(d\underline{\omega})$ of the homogeneous field $X(\underline{x})$ is given by

$$\begin{aligned} \text{Cum}(Z(d\underline{\omega}_1), Z(d\underline{\omega}_2), Z(d\underline{\omega}_3)) &= \delta(\Sigma_1^3 \underline{\omega}_k) S_3(\underline{\omega}_1, \underline{\omega}_2, \underline{\omega}_3) d\underline{\omega}_1 d\underline{\omega}_2 d\underline{\omega}_3 \\ &= \delta(\Sigma_1^3 \rho_k \hat{\underline{\omega}}_k) S_3(\rho_1, \rho_2, \rho_3) \prod_{k=1}^3 \Omega(d\hat{\underline{\omega}}_k) \rho_k d\rho_k, \end{aligned}$$

where $\hat{\underline{\omega}}_k = \underline{\omega}_k / |\underline{\omega}_k|$. The stochastic spectral measures $Z_{\ell}(\rho d\rho)$ are connected to $Z(d\underline{\omega})$ by

$$Z_{\ell}(\rho d\rho) = \int_0^{2\pi} i^{\ell} e^{-i\ell \eta} Z(d\eta \rho d\rho),$$

therefore

$$\begin{aligned} \text{Cum}(Z_0(\rho_1 d\rho_1), Z_{\ell}(\rho_2 d\rho_2), Z_{-\ell}(\rho_3 d\rho_3)) &= S_3(\rho_1, \rho_2, \rho_3) \\ &\quad \times \int_0^{2\pi} \int_0^{2\pi} e^{-i\ell(\eta_3 - \eta_2)} \delta(\Sigma_1^3 \rho_k \hat{\underline{\omega}}_k) \prod_{k=1}^3 \rho_k d\rho_k d\eta_k. \end{aligned} \quad (5)$$

In order to understand the applicability of the Dirac 'function' in polar coordinates we express it by an integral through the Jacobi-Anger expansion (2) and obtain

$$\begin{aligned} \delta(\Sigma_1^3 \rho_k \hat{\underline{\omega}}_k) &= \frac{1}{(2\pi)^2} \iint_{\mathbb{R}^2} e^{i(\underline{\lambda} \cdot \Sigma_1^3 \underline{\omega}_k)} d\underline{\lambda} \\ &= \frac{1}{(2\pi)^2} \int_0^{\infty} \int_0^{2\pi} \prod_{k=1}^3 \sum_{\ell_k = -\infty}^{\infty} i^{\ell_k} J_{\ell_k}(\rho_k \lambda) e^{i\ell_k(\eta_k - \xi)} \lambda d\lambda d\xi. \end{aligned} \quad (6)$$

Now substitute. (6) into 5. The 'spherical harmonics' are orthogonal

$$\int_0^{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-i\ell(\eta_3 - \eta_2)} \prod_{k=1}^3 e^{i\ell_k(\eta_k - \xi)} d\eta_k d\xi = \delta_{\ell_1} \delta_{\ell_2 + \ell} \delta_{\ell_3 - \ell} (2\pi)^4,$$

hence obtain from the equations

$$\int_0^{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-i\ell(\eta_3 - \eta_2)} \delta(\Sigma_1^3 \rho_k \hat{\underline{\omega}}_k) d\eta_1 d\eta_2 d\eta_3 = (2\pi)^2 \int_0^{\infty} J_0(\rho_1 \lambda) J_{-\ell}(\rho_2 \lambda) J_{\ell}(\rho_3 \lambda) \lambda d\lambda.$$

This integral can be evaluated, if $|\rho_2 - \rho_3| < \rho_1 < \rho_2 + \rho_3$, and let us denote $R = (\rho_2^2 + \rho_3^2 - \rho_1^2) / (2\rho_2 \rho_3)$, $\rho_1^2 = \rho_2^2 + \rho_3^2 - 2\rho_2 \rho_3 \cos(\eta)$, then

$$\int_0^{\infty} J_0(\rho_1 \lambda) J_{\ell}(\rho_2 \lambda) J_{\ell}(\rho_3 \lambda) \lambda d\lambda = \frac{\cos(\ell \arccos(R))}{\pi \rho_2 \rho_3 \sqrt{1 - R^2}},$$

see [PBM86] Tom. II, 2.12.41.16, hence

$$\int_0^{2\pi} \int \int e^{-i\ell(\eta_1 - \eta_2)} \delta(\Sigma_1^3 \rho_k \hat{\omega}_k) d\eta_1 d\eta_2 d\eta_3 = 4\pi (-1)^\ell \frac{\cos(\ell \arccos(R))}{\rho_2 \rho_3 \sqrt{1 - R^2}}.$$

Since $J_\ell = (-1)^\ell J_{-\ell}$ we have

$$\begin{aligned} \text{Cum}(Z_0(\rho_1 d\rho_1), Z_\ell(\rho_2 d\rho_2), Z_{-\ell}(\rho_3 d\rho_3)) \\ = 4\pi (-1)^\ell \delta(\rho\Delta) \frac{\cos(\ell \arccos(R))}{\rho_2 \rho_3 \sqrt{1 - R^2}} S_3(\rho_1, \rho_2, \rho_3) \prod_{k=1}^3 \rho_k d\rho_k \end{aligned}$$

where $\delta(\rho\Delta) = \delta(\rho_2^2 + \rho_3^2 - 2\rho_2\rho_3 \cos \eta - \rho_1^2)$ and the wave numbers ρ_1 , ρ_2 , and ρ_3 should satisfy the triangle relation. Now we arrive at the following expression

$$\begin{aligned} \text{Cum}(X(0), X(r_2 N), X(\underline{x}_3)) &= 4\pi \sum_{\ell=-\infty}^{\infty} \int \int \int_0^{\infty} e^{-i\ell\varphi} J_\ell(\rho_2 r_2) J_{-\ell}(\rho_3 r_3) \\ &\quad \times (-1)^\ell \delta(\rho\Delta) \frac{\cos(\ell \arccos(R))}{\rho_2 \rho_3 \sqrt{1 - R^2}} S_3(\rho_1, \rho_2, \rho_3) \prod_{k=1}^3 \rho_k d\rho_k \\ &= 4\pi \sum_{\ell=-\infty}^{\infty} \int \int_0^{\infty} \int_{|\rho_2 - \rho_3|}^{\rho_2 + \rho_3} e^{-i\ell\varphi} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \\ &\quad \times \frac{\cos(\ell \arccos(R))}{\rho_2 \rho_3 \sqrt{1 - R^2}} S_3(\rho_1, \rho_2, \rho_3) \prod_{k=1}^3 \rho_k d\rho_k \\ &= 4\pi \sum_{\ell=-\infty}^{\infty} \int \int_0^{\infty} \int_0^{\pi} e^{-i\ell\varphi} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \\ &\quad \times \frac{\cos(\ell\eta)}{\sqrt{1 - \cos^2(\eta)}} S_3(\rho_1, \rho_2, \rho_3) \sin \eta d\eta \prod_{k=2}^3 \rho_k d\rho_k \\ &= 4\pi \sum_{\ell=-\infty}^{\infty} \int \int_0^{\infty} \int_0^{\pi} e^{-i\ell\varphi} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \\ &\quad \times \cos(\ell\eta) S_3(\rho_1, \rho_2, \rho_3) d\eta \prod_{k=2}^3 \rho_k d\rho_k, \end{aligned}$$

where $\rho_1^2 = \rho_2^2 + \rho_3^2 - 2\rho_2\rho_3 \cos(\eta)$ and

$$\frac{\rho_1 d\rho_1}{d\eta} = \rho_2 \rho_3 \sin \eta.$$

The function

$$\begin{aligned} \mathcal{T}(\varphi, \rho_2, \rho_3 | \eta, r_2, r_3) &= \sum_{\ell=-\infty}^{\infty} e^{-i\ell\varphi} J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell\eta) \\ &= \sum_{\ell=-\infty}^{\infty} \cos(\ell\varphi) J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell\eta) \\ &= J_0(\rho_2 r_2) J_0(\rho_3 r_3) + 2 \sum_{\ell=1}^{\infty} \cos(\ell\varphi) J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell\eta), \end{aligned} \quad (7)$$

gives the transformation of the bispectrum $S_3(\rho_1, \rho_2, \rho_3)$ into the bicovariance $\mathcal{C}_3(r_1, r_2, r_3)$ since both angles φ and η together with two sides define the third sides ρ_1 and r_1 of the triangles, given by wave numbers (ρ_1, ρ_2, ρ_3) and distances (r_1, r_2, r_3) . Notice that the transformation \mathcal{T} can be simplified

$$\begin{aligned} \mathcal{T}(\varphi, \rho_2, \rho_3 | \eta, r_2, r_3) &= \frac{1}{2} \left(J_0(\rho_2 r_2) J_0(\rho_3 r_3) + 2 \sum_{\ell=1}^{\infty} J_{\ell}(\rho_2 r_2) J_{\ell}(\rho_3 r_3) \cos(\ell(\eta + \varphi)) \right) \\ &\quad + \frac{1}{2} \left(J_0(\rho_2 r_2) J_0(\rho_3 r_3) + 2 \sum_{\ell=1}^{\infty} J_{\ell}(\rho_2 r_2) J_{\ell}(\rho_3 r_3) \cos(\ell(\eta - \varphi)) \right) \\ &= \frac{1}{2} (J_0(w_+) + J_0(w_-)), \end{aligned}$$

where $w_+ = \sqrt{(\rho_2 r_2)^2 + (\rho_3 r_3)^2 - 2\rho_2 r_2 \rho_3 r_3 \cos((\eta + \varphi))}$,

$w_- = \sqrt{(\rho_2 r_2)^2 + (\rho_3 r_3)^2 - 2\rho_2 r_2 \rho_3 r_3 \cos((\eta - \varphi))}$, see [EMOT54] Tom.2, Ch7.7.15. for

$$\begin{aligned} J_0(z) J_0(Z) + 2 \sum_{\ell=1}^{\infty} J_{\ell}(z) J_{\ell}(Z) \cos(\ell\vartheta) &= J_0(w) \\ w &= \sqrt{z^2 + Z^2 - 2zZ \cos(\vartheta)} \end{aligned}$$

$\underline{\omega}_1 + \underline{\omega}_2 + \underline{\omega}_3 = 0$, hence the wave numbers ρ_1 , ρ_2 , and ρ_3 satisfy the triangle relation (they should be able to form a triangle).

We have now established a relationship between the bicovariance and the bispectrum.

$$\text{Cum}(X(0), X(r_2 N), X(\underline{x}_3)) = 2\pi \int_0^{\infty} \int_0^{\pi} (J_0(w_+) + J_0(w_-)) S_3(\rho_1, \rho_2, \rho_3) d\eta \prod_{k=2}^3 \rho_k d\rho_k.$$

We show that the bispectrum can be also expressed by the bicovariance in turn. The bicovariance $\text{Cum}(X(0), X(\underline{x}_2), X(\underline{x}_3))$ depends on the length of vectors \underline{x}_2 , \underline{x}_3 and the angle φ between them, this way a triangle with vertices 0, \underline{x}_2 , \underline{x}_3 is formed with length of the third side r_1 , such that $r_1^2 = r_2^2 + r_3^2 - 2r_2 r_3 \cos(\varphi)$. According to this definition of r_1 , we introduce the notation

$$\mathcal{C}(r_1, r_2, r_3) = \text{Cum}(X(0), X(\underline{x}_2), X(\underline{x}_3)).$$

We show that

$$S_3(\rho_1, \rho_2, \rho_3) = \frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{\pi} (J_0(w_+) + J_0(w_-)) \mathcal{C}(r_1, r_2, r_3) d\varphi \prod_{k=2}^3 r_k dr_k,$$

where the earlier notations are used.

Consider the integral

$$\begin{aligned} I(\rho_1, \rho_2, \rho_3 | \kappa_1, \kappa_2, \kappa_3) &= \int_0^{\infty} \int_0^{\pi} \int_0^{\pi} \left(J_0(\rho_2 r_2) J_0(\rho_3 r_3) + 2 \sum_{\ell=1}^{\infty} \cos(\ell\varphi) J_{\ell}(\rho_2 r_2) J_{\ell}(\rho_3 r_3) \cos(\ell\eta) \right) \\ &\quad \times \left(J_0(\kappa_2 r_2) J_0(\kappa_3 r_3) + 2 \sum_{\ell=1}^{\infty} \cos(\ell\varphi) J_{\ell}(\kappa_2 r_2) J_{\ell}(\kappa_3 r_3) \cos(\ell\vartheta) \right) d\varphi \prod_{k=2}^3 r_k dr_k, \end{aligned}$$

notice first that $\cos(\ell\varphi)$ is an orthogonal system on $[0, \pi]$, i.e.

$$\int_0^{\pi} \cos(\ell_1 \varphi) \cos(\ell_2 \varphi) d\varphi = \delta_{\ell_1=\ell_2} \begin{cases} \pi & \text{if } \ell_1 = 0 \\ \frac{\pi}{2} & \text{if } \ell_1 \neq 0 \end{cases},$$

hence the integral by φ gives

$$\begin{aligned} I(\rho_1, \rho_2, \rho_3 | \kappa_1, \kappa_2, \kappa_3) &= \pi \int_0^\infty J_0(\rho_2 r_2) J_0(\kappa_2 r_2) r_2 dr_2 \int_0^\infty J_0(\rho_3 r_3) J_0(\kappa_3 r_3) r_3 dr_3 \\ &\quad + 2\pi \sum_{\ell=1}^\infty \cos(\ell\eta) \cos(\ell\vartheta) \int_0^\infty J_\ell(\rho_2 r_2) J_\ell(\kappa_2 r_2) r_2 dr_2 \int_0^\infty J_\ell(\rho_3 r_3) J_\ell(\kappa_3 r_3) r_3 dr_3. \end{aligned}$$

The integral of Bessel functions provides the Dirac function

$$\int_0^\infty J_\ell(\rho r) J_\ell(\kappa r) r dr = \frac{\delta(\rho - \kappa)}{\rho},$$

see [AW01] Sect 11. therefore

$$\begin{aligned} I(\rho_1, \rho_2, \rho_3 | \kappa_1, \kappa_2, \kappa_3) &= \pi^2 \delta(\rho_2 - \kappa_2) \delta(\rho_3 - \kappa_3) \left(\frac{1}{\pi} + \frac{2}{\pi} \sum_{\ell=1}^\infty \cos(\ell\eta) \cos(\ell\vartheta) \right) \\ &= \pi^2 \delta(\rho_2 - \kappa_2) \delta(\rho_3 - \kappa_3) \delta(\eta - \vartheta), \end{aligned}$$

since the $\frac{1}{\pi}$ and $\sqrt{\pi/2} \cos(\ell\eta)$ forms an orthonormal system. see [AW01] Sect.11.

Theorem 2 *Let us assume that both integral exists below then*

$$\begin{aligned} \mathcal{C}(r_1, r_2, r_3) &= 2\pi \int_0^\infty \int_0^\pi (J_0(w_+) + J_0(w_-)) S_3(\rho_1, \rho_2, \rho_3) d\eta \prod_{k=2}^3 \rho_k d\rho_k, \\ S_3(\rho_1, \rho_2, \rho_3) &= \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi (J_0(w_+) + J_0(w_-)) \mathcal{C}(r_1, r_2, r_3) d\varphi \prod_{k=2}^3 r_k dr_k. \end{aligned}$$

Note that in some cases it is more convenient to use the transformation (7) to obtain

$$\mathcal{C}(r_1, r_2, r_3) = 4\pi \int_0^\infty \int_0^\pi \mathcal{T}(\varphi, \rho_2, \rho_3 | \eta, r_2, r_3) S_3(\rho_1, \rho_2, \rho_3) d\eta \prod_{k=2}^3 \rho_k d\rho_k, \quad (8)$$

$$S_3(\rho_1, \rho_2, \rho_3) = \frac{1}{4\pi^3} \int_0^\infty \int_0^\pi \mathcal{T}(\varphi, \rho_2, \rho_3 | \eta, r_2, r_3) \mathcal{C}(r_1, r_2, r_3) d\varphi \prod_{k=2}^3 r_k dr_k. \quad (9)$$

A very useful model with finite many parameters on the plane is connected to the Laplacian operator.

Example 3 *The spatial white noise $\partial W(r, \varphi)$, on the plane is given as a generalized field by the series representation*

$$\partial W(r, \varphi) = \sum_{\ell=-\infty}^{\infty} e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) W_\ell(\rho d\rho),$$

where $W_\ell(\rho d\rho)$ with $E|W_\ell(\rho d\rho)|^2 = \sigma^2 \rho d\rho$, see [Yag87], [Yad83]. We consider the Laplacian field on the plane by the equation

$$(\nabla^2 - c^2) X(r, \varphi) = \partial W(r, \varphi), \quad (10)$$

where

$$\nabla^2 = \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2},$$

is the Laplacian operator in terms of spherical coordinates. Now we have

$$\begin{aligned} \nabla^2 e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho) &= e^{i\ell\varphi} \int_0^\infty \left[\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\ell^2}{r^2} \right] J_\ell(\rho r) Z_\ell(\rho d\rho) \\ &= -e^{i\ell\varphi} \int_0^\infty \rho^2 J_\ell(\rho r) Z_\ell(\rho d\rho), \end{aligned}$$

hence

$$(\nabla^2 - c^2) e^{i\ell\varphi} \int_0^\infty J_\ell(\rho r) Z_\ell(\rho d\rho) = -e^{i\ell\varphi} \int_0^\infty (\rho^2 + c^2) J_\ell(\rho r) Z_\ell(\rho d\rho).$$

Let us compare the two sides of the equation (10), and obtain

$$\begin{aligned} -(\rho^2 + c^2) Z_\ell(\rho d\rho) &= W_\ell(\rho d\rho), \\ (\rho^2 + c^2)^2 2\pi F(\rho d\rho) &= \sigma^2 \rho d\rho, \\ F(\rho d\rho) &= \frac{\sigma^2 \rho d\rho}{2\pi (\rho^2 + c^2)^2}. \end{aligned}$$

We have got the covariance

$$\begin{aligned} \text{Cov}(X(\underline{x}), X(\underline{y})) &= 2\pi \int_0^\infty J_0(\rho r) F(\rho d\rho) \\ &= \int_0^\infty J_0(\rho r) \frac{\sigma^2 \rho d\rho}{(\rho^2 + c^2)^2} \\ &= \sigma^2 \frac{r K_{-1}(cr)}{2c}. \end{aligned}$$

This covariance is of Matérn Class, see [Whi54] ($K_{-1} = K_1$), in terms of modified Bessel (Hankel) function. The Bispectrum of the Laplacian model follows

$$S_3(\rho_1, \rho_2, \rho_3) = \prod_{k=1}^3 \frac{\sigma^2}{\rho_k^2 + c^2},$$

where $\rho_1^2 = \rho_2^2 + \rho_3^2 - 2\rho_2\rho_3 \cos(\eta)$. We apply the transformation (7)

$$\mathcal{T}(\rho_1, \rho_2, \rho_3 | \kappa_1, \kappa_2, \kappa_3) = J_0(\rho_2 r_2) J_0(\rho_3 r_3) + 2 \sum_{\ell=1}^{\infty} \cos(\ell\varphi) J_\ell(\rho_2 r_2) J_\ell(\rho_3 r_3) \cos(\ell\eta),$$

for establishing the connection to the cumulants and simplifying the bicovariance. If $(a/b)^2 < 1$, we have

$$\int_0^\pi \frac{\cos(\ell\eta)}{b + a \cos(\eta)} d\eta = \frac{\pi}{\sqrt{b^2 - a^2}} \left(\frac{\sqrt{b^2 - a^2} - b}{a} \right)^\ell,$$

see [GR00] 3.613.1. Put

$$\begin{aligned} a &= -2\rho_2\rho_3, \quad b = \rho_2^2 + \rho_3^2 + c^2 \\ b^2 - a^2 &= \left((\rho_2 - \rho_3)^2 + c^2 \right) \left((\rho_2 + \rho_3)^2 + c^2 \right), \end{aligned}$$

hence

$$\int_0^\pi \frac{\cos(\ell\eta)}{\rho_2^2 + \rho_3^2 - 2\rho_2\rho_3 \cos(\eta) + c^2} d\eta = \frac{\pi}{\sqrt{b^2 - a^2}} \left(\frac{\sqrt{b^2 - a^2} - b}{a} \right)^\ell.$$

In this way we arrive to the Fourier expansion of the bicovariances

$$\mathcal{C}(r_1, r_2, r_3) = f_0 + 2 \sum_{\ell=1}^{\infty} f_{\ell} \cos(\ell\varphi),$$

with coefficients

$$f_{\ell} = 4\pi \int_0^{\infty} \int \frac{\pi}{\sqrt{b^2 - a^2}} \left(\frac{\sqrt{b^2 - a^2} - b}{a} \right)^{\ell} J_{\ell}(\rho_2 r_2) J_{\ell}(\rho_3 r_3) \prod_{k=2}^3 \rho_k d\rho_k.$$

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